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Abstract
We use trichord subsets to discuss Z-related set classes, and prove an analogue of Patterson's First Theorem involving the difference between the multiplicities of trichord subsets in a Z-pair and its complement.

1 Trichordal substructure as a refinement of dyad substructure

The following suggestive definition of the interval vector of a pitch-class set (henceforth PCset) motivates the following discussion:

Definition 1. *The interval vector of a PCset C -call it IV(C)- is the unique 6- tuple of integers whose m-th entry is the number of unique subsets of C that are members of the set-class 2-m.*

Here, any set class with the label *2-i* simply corresponds to pairs of distinct pitch-classes that are *i* half-steps apart. Analogously,

Definition 2. *The trichord vector of a PCset C -call it TV(C)- is the unique 12-tuple of integers whose m-th entry is the number of unique subsets of C corresponding to the set-class with label 3-m.*

As an example, the PCset {0146} has the interval vector <111111> and the trichord vector <001010110000>. The first vector indicates that one of each dyad SC appears in the PCset, and the second indicates that the trichordal SCs 3-3,3-5,3-7, and 3-8 each appear once as subsets of our PCset. Likewise, the PCset {0137} has the interval vector <111111> and the trichord vector <010010010010> (in this case the trichord set classes are (013),(016),(026), and (037)).

The trichordal substructure of PCsets yields a strictly finer system of equivalence than the dyad substructure. We say 'finer' because the information of an interval vector can be obtained from a trichord vector as follows:

Lemma 1. *For any PCset C of size n,* $IV(C)=\frac{1}{(n-2)}\sum_{\{x,y,z\}\subseteq C} IV(\{x,y,z\})$

> *Proof: Each dyad in a chord of size n appears in exactly* $n-2$ *trichords (a dyad appears in each trichord that contains both of its 2 elements, and in a chord of size* n *there are* $n-2$ *candidates for the third element of such a trichord) so a linear mapping that sends each trichord to its corresponding interval vector will map each chord of size* n *to its interval vector with each entry multiplied by* $n-2$.

There is (as the proof is worded to suggest) a way to express this equivalence with vectors and matrices. Letting $C_{m\text{-}n}$ denote the number of instances in which the set class with label m-n appears as a subset of the PCset C, we can rewrite Lemma 1 to give the following formula. Note that each *i-th* column of the 6X12 matrix below is nothing other than the interval vector of the set class *3-i*.

Corollary 1.

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_{3-1} \\ c_{3-2} \\ c_{3-3} \\ c_{3-4} \\ c_{3-5} \\ c_{3-6} \\ c_{3-7} \\ c_{3-8} \\ c_{3-9} \\ c_{3-10} \\ c_{3-11} \\ c_{3-12} \end{bmatrix} = (n-2) \begin{bmatrix} c_{2-1} \\ c_{2-2} \\ c_{2-3} \\ c_{2-4} \\ c_{2-5} \\ c_{2-6} \end{bmatrix}$$

And we say that trichordal class vector (henceforth TCV) equivalence is 'strictly finer' than ICV equivalence because there exist z-pairs whose members have dissimilar trichord vectors (Table A). In fact, the members of *every* z-pair have dissimilar trichord vectors. As a result of this, every pair of symmetry nonequivalent PCsets (up to translation and inversion) have nonequivalent trichord vectors, and the equivalence between PCsets formed by trichord vectors is none other than equivalence under translation and inversion.

By speaking of equivalence via trichord vectors we want to compare the equivalence classes of PCsets under symmetry with the coarser equivalence classes generated by their interval vectors in a way that allows us to describe the explanatory gap between them (namely, z-pairs).

2 Trichord difference and M_5

If we are interested in discussing the difference in the trichord vectors of a given z-pair, we should note that such a difference (literally; one trichord vector minus the other, $TCV(A) - TCV(B)$ where A and B are any two members of a z-pair) lies in the *null-space*[1] of the 6X12 matrix in Corollary 1. This is a 6-dimensional subspace, and a basis of it is given below:

<1000-2-2021000> <0100-1-2120-100>
<0010-1-1020-11-1> <0002-2-102000-1>
<10-200000-1020> <01-1000-100010>

1First: think of that matrix as a linear transformation φ from a 12-dimensional vector space to a 6-dimensional vector space (let's call them V and W respectively). Formally speaking, the null-space of that linear transformation $\varphi : V \rightarrow W$ is the set of all elements $v \in V$ for which $\varphi(v) = 0$. The difference of two trichord vectors can be thought of as an element of the same vector space, (i.e. we can sensibly talk about what φ does to this 'difference vector') and $\varphi(v) = 0$ means that the sum of 'additions' and 'removals' of trichords expressed by the difference vector has no net effect on the interval vector.

It has been known for some time that the M-operation, that is multiplication of pitch-class integers by 5, has a close and intriguing connection to Z-related sets.[2] The top four vectors generate the subspace of this null-space formed by vectors that are symmetric under M_5, i.e. $M5(\vec{v})=\vec{v}$. The bottom two vectors generate the subspace of vectors that are anti-symmetric under M_5 (i.e. $M5(\vec{v})=-\vec{v}$). All but eight Z pairs have either an M_5 symmetric or M_5 antisymmetric Trichord Class Difference Vector, or TCDV. A detailed treatment of these phenomena is given in Appendix A.

An important early result in musical set theory was the hexachordal theorem, which states that two complementary hexachords are necessarily Z-related, that is they must share the same ICV. It is now known that a similar result was proven much earlier by crystallographer Lindo Patterson.[3] An important generalization of this theorem was given in 2009 by Ballinger, Benbernou,Martin, O'Rourke, and Toussaint, and we use their continuous hexachordal theorem to show a generalization to TCVs.[4] If we use f to denote the characteristic function of a PCset R (i.e. $f(x)=1$ if $x\in R$ and $f(x)=0$ otherwise) we can represent the number of instances of an interval of size d for any PCset R in base n as follows:

$$H_d(R)=\sum_{x=1}^{n} f(x)f(x+d)$$

There is one caveat here: in an even base where $n=2d$, intervals of size d will be counted twice. Similarly, the way we are about to express coefficients in a TCV will overcount the instances of [4; 4; 4] by a factor of three. But as long as we overcount things consistently, then our proof will remain valid.

$$H_{(0,c,d)}(R)=\sum_{x=1}^{n} f(x)f(x+c)f(x+d)$$

We can prove the hexachordal theorem easily by noting that the value of $f(x)$ for the complement of R (let's write them $\overline{R}$ and $\overline{f(x)}$) is 0 when $f(x)=1$ and vice verse. So we can use the identity $\overline{f(x)}=1-f(x)$ to re-prove the classic result:

Lemma 2 (Hexachordal Theorem). *For a distance d and PCset R in modulus n,* $H_d(\overline{R})=H_d(R)+n-2|R|$, *where* $|R|$ *denotes the cardinality of R*

Proof:

$$\begin{aligned}\sum_{x=1}^{n} \overline{f(x)f(x+d)} &= \sum_{x=1}^{n}(1-f(x))(1-f(x+d))\\ &= \sum_{x=1}^{n} 1-f(x)-f(x+d)+f(x)f(x+d)\\ &= \sum_{x=1}^{n} 1-\sum_{x=1}^{n} f(x)-\sum_{x=1}^{n} f(x+d)+\sum_{x=1}^{n} f(x)f(x+d)\\ &= H_d(R)+n-2|R|\end{aligned}$$

2 See Capuzzo, 2008 for a useful overview

3 Patterson 1944

4 Brad Ballinger, Nadia Benbernou, Francisco Gomez-Martin, J. O'Rourke, Godfried Toussaint, 2009

The obvious corollaries are that for any pair of PCsets *R* and *S* of the same cardinality, $ICV(S) - ICV(R) = ICV(\overline{S}) - ICV(\overline{R})$. Thus if *R* and *S* are both z-related, their complements will be z-related as well. Analogously,

Theorem 1: *For a trichord [0,c,d] and PCset R in modulus n,*

$$H_{[0,c,d]}(\overline{R}) = -H_{[0,c,d]}(R) + H_d(R) + H_c(R) + H_{d-c} - 3|R| + n$$

Proof:

$$\begin{aligned}\sum_{x=1}^{n} \overline{f(x) f(x+c) f(x+d)} &= \sum_{x=1}^{n} (1 - f(x))(1 - f(x+c))(1 - f(x+d)) \\ &= \sum_{x=1}^{n} 1 - f(x) - f(x+c) - f(x+d) - f(x) f(x+c) \\ &\quad + f(x+c) f(x+d) + f(x) f(x+d) - f(x) f(x+c) f(x+d) \\ &= n - 3|R| + H_c(R) + H_{d-c}(R) + H_d(R) - H_{[0,c,d]}(R)\end{aligned}$$

The relevant application is that for any z-pair *R* and *S*, that is $H_d(R) = H_d(S)$ for all *d*, then $H_{[0,c,d]}(R) - H_{[0,c,d]}(S) = H_{[0,c,d]}(\overline{S}) - H_{[0,c,d]}(\overline{R})$. That is to say the *trichordal difference* is inverted in complementary Z-pairs.

Should work with higher subsets become necessary, the logic generalizes naturally- if *R* and *S* have identical trichord vectors, their tetrachordal difference is preserved by complementation. If *R* and *S* are tetrachordally equivalent, their pentachordal difference is inverted by complementation, etc., etc.

Appendix A

Two musical examples examining M_5 symmetric/antisymmetric trichord class difference vectors of z-related set classes[1]

Case 1. *M_5 antisymmetry*

Every Z-related pair with a M_5 *antisymmetric* trichord class difference vector must, under M_5, map to the other member of the pair.[2]

Let us examine a familiar case, the two all interval, Z-related tetrachords, 4-15 and 4-29, call them A and B.

A= {0146}
B= {0137}

ICV(A)= <111111>
ICV(B)= <111111>

However, their trichordal substructures differ.

trichord	# in A	# in B
3-1	0	0
3-2	0	1
3-3	1	0
3-4	0	0
3-5	1	1
3-6	0	0
3-7	1	0
3-8	1	1
3-9	0	0
3-10	0	0
3-11	0	1
3-12	0	0

---

1 The musical examples were composed to exemplify and examine the trichordal substructure of a given SC and its transformation under M_5. All are in two part counterpoint; every trichordal subset of a given SC occurs as a linear melodic segment. This gives each trichordal subset a temporal locality, hopefully rendering M_5 transformations, which change trichordal substructure, better heard. With an aim toward clarity, the examples interpret SCs as prime forms, that is 0=0 when going from IC to PC space.

2 That is, for any Z-pair (A, B), if their TCDV is M_5 *antisymmetric*, that is is $M_5(\overrightarrow{TCDV(A,B)}) = -\overrightarrow{TCDV(A,B)}$, then $M_5(A)=B$ and $M_5(B)=A$

Correspondingly, for any M_5 *symmetric* TCDV, that is $M_5(\overrightarrow{TCDV(A,B)}) = \overrightarrow{TCDV(A,B)}$, then $M_5(A)=A$ and $M_5(B)=B$

We express this information as a trichord class vector, or TCV.

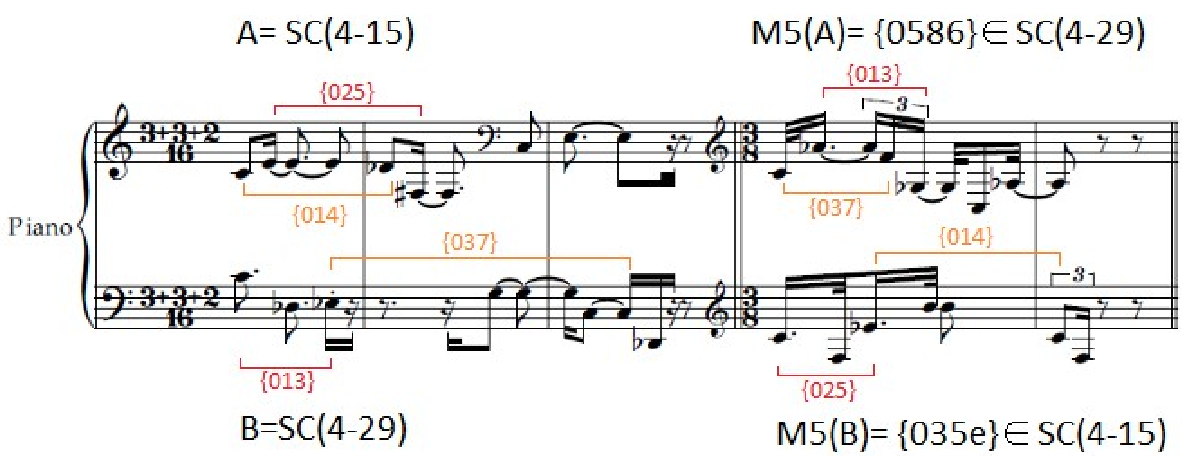
TCV(A)= <001010110000>
TCV(B)= <010010010010>

To compare the two TCVs in a more handy way, we take their difference, $TCV(A)-TCV(B)$ The result of this we call the *trichord class difference vector*, or TCDV for short. We draw arrows to indicate M_5 variant trichordal SCs.

$$TCDV(A,B)=\langle 0\;-1\;1\;0\;0\;0\;1\;0\;0\;0\;-1\;0\rangle$$

We see that in this case, A lacks all and only those M_5 variant trichordal SCs that B possesses, and vice verse. The rest of the entries are either zero because they are absent from both A and B, as with their mutually lacking (012), or zero because their values canceled upon subtraction, meaning A and B had the same amount of that trichord, as with their shared (016). (See **Example 1**)

Example 1.



Case 2. *M_5 symmetry*

Every Z-related pair with a M_5 *symmetric* TCDV must, under M_5, map to itself.

*C*onsider the Z-related pentachords 5-12 and 5-36; call them X and Y.

X= {01356}

Y= {01247}

We again wish to investigate the trichordal subset structure for both sets as TCVs.

TCV(X)= <020221200100>
TCV(Y)= <111021101110>

Now we examine the difference of these two trichord class vectors, $TCV(X)-TCV(Y)$, which gives the following TCDV as a result.

$$TCDV(X,Y)=<-1\,1\,-1\,2\,0\,0\,1\,0\,-1\,0\,-1\,0>$$

Again we note that for any TCDV, a negative entry indicates that that trichord was either not present in X, or present in greater numbers in Y. Reciprocally we also observe that a positive entry indicates either a trichordal SC that X possessed and Y did not, or one which at any rate X had more of. We draw arrows indicating M_5 variant entries.

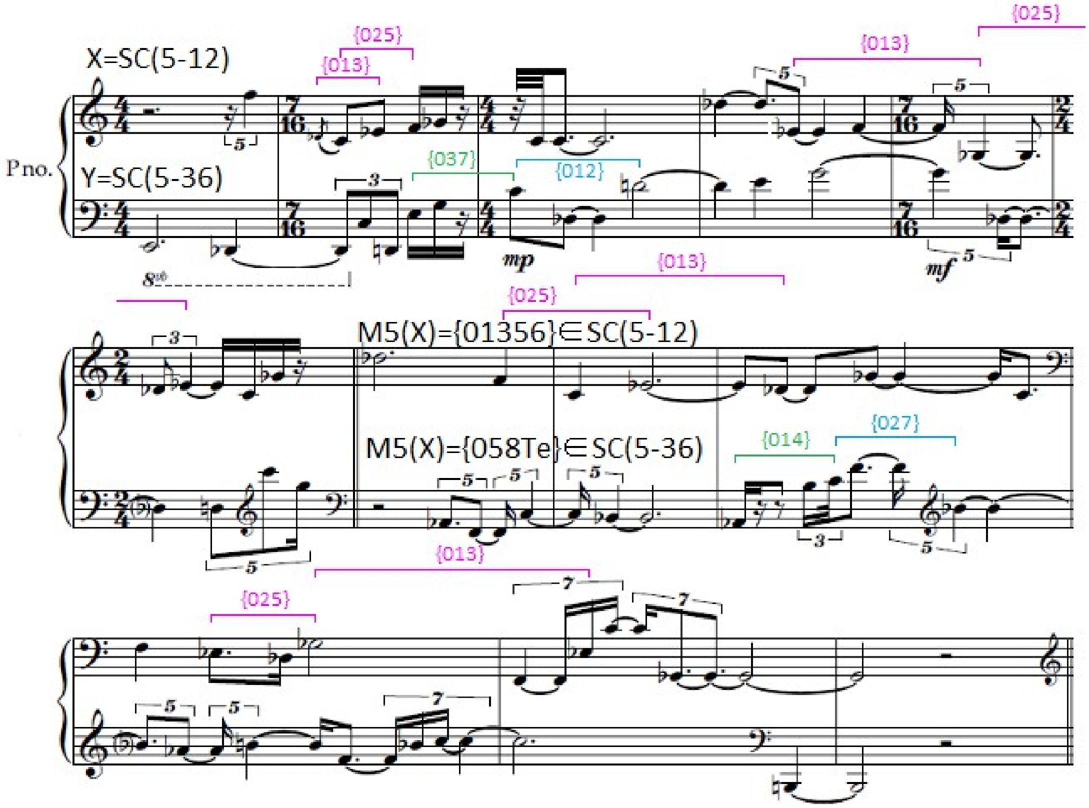
$$TCDV(X,Y)=<-1\,1\,-1\,2\,0\,0\,1\,0\,-1\,0\,-1\,0>$$

The blue arrow shows the transformation of the (012) trichord, present only in Y, to the (027) trichord, also exclusive to Y.

The green arrow represents the transformation from (014} → (037). Both trichords are, also, exclusive to Y.

The pink arrow represents the transformation of the (013) trichord to the (025). Examining this case in more detail, we see that both X and Y have SCs 3-2 and 3-7 as subsets, but X has two of each, and Y just one.

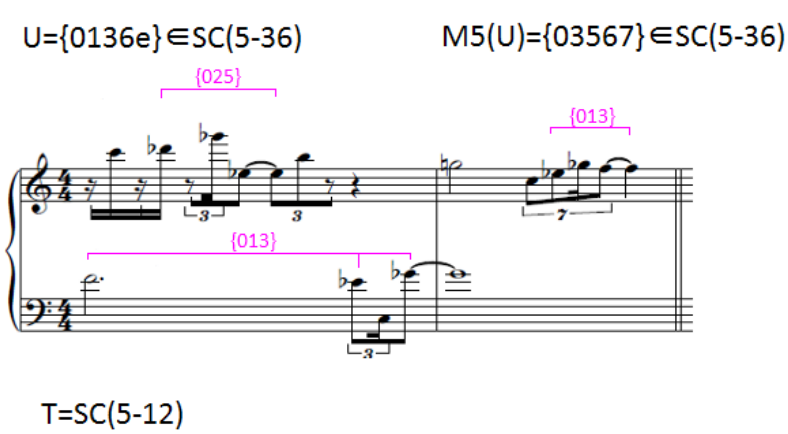
Example 2.



Any one of these observations alone is enough to prove that there is no series of operations up to and including M_5 that can transform these two SCs into one another.

As an aside, it is interesting to note that the TCDV for this pair seems to leave open the question of whether M_5 will necessarily always exclude these sets' entire subset structures in PC space, or if at least one 3-2 → 3-7 transformation can be shared between X and Y. It turns out both cases are possible. For proof, consider the PCsets {01356} and {058Te}, which exclude all their M_5 transformations from one another (**Example 2**). Contrast this with PCsets {01356} and {03567}, which hold the one possible M_5 3-7 transformation through common tones (M_5({356}) → {316}; see **Example 3**).

Example 3.



Summary:

Z-pairs are on a spectrum of connectedness, with the most connected being M_5 antisymmetric pairs, which attain bijective M_5 variant trichord subset mapping, as in **example 1**. There is no instance of *total* abstract *exclusion* of all M_5 variant trichordal subsets for M_5 symmetric pairs, but pairs with a comparatively high number of such exclusions are the least connected, as in **example 2**. In the middle are the eight z-pairs which have neither M_5 symmetric nor M_5 antisymmetric TCDVs.

A precise measure of this spectrum is given by the following formula:

$$\frac{\sum_{x \in I} |EMB(x, A) - EMB(M(x), B)|}{\sum_{x \in I} |EMB(x, A) + EMB(M(x), B)|}$$

Where I is the indexing set of M_5-variant trichords, x is some member of that indexing set, and A and B are the two set-classes whose M-relatedness we are testing. The EMB function, due to Lewin, takes two sets as arguments, EMB(x,A), and counts the number of members of x in A.[3]

The formula counts how many versions of $M_5(x)$ exist in set-class B, subtracts that from the total number of forms of x in A, and then takes the absolute value of this difference, which tells us how many of the differences between the two TCDVs for that trichord were abstractly due to M_5 transformations. Taking the absolute value of this calculation has the effect of making it irrelevant which member of a Z-pair we label 'A' and which 'B'; the difference result will now be the same no matter what order we place them in the formula. We then compute these differences for every value of x in the indexing set, and sum those differences, which gives us the total number of trichordal transformations between A and B abstractly due to M_5. An important step is then to divide this result by some factor that de-privileges the relatively high values a larger A or B will achieve unjustly; sets of higher cardinality naturally have many more subset transformations possible, and so dividing by the sum of all the relevant embeddings in both A and B normalizes the result so that the highest possible value of relatedness is always unity, no matter the cardinality of A or B.

Calculating these values for every Z-related pair yields the strength values in the last column of Table 1. As expected the M_5 antisymmetric pairs all have a M_5 relatedness measure of 1, with the remaining pairs lying somewhere between 1.0 and 0.6.

3 See Lewin 1979 and Rahn 1979 for an important discussion where these ideas are explored in some depth.

	$M_5(A)=A$, $M_5(B)=B$ (M_5 symmetric)	**$M_5(A)=B$, $M_5(B)=A$** (M_5 antisymmetric)	**$M_5(A_1)=A_2$, $M_5(B_1)=(B_2)$**	**$M_5(A_1)=B_2$, $M_5(B_1)=(A_2)$**	**TCV(A)**	**TCV(B)**	**TCV(A)-TCV(B)**	**Strength Indicator** $\frac{\sum_{x\in I}\lvert EMB(x,A)-EMB(M(x),B)\rvert}{\sum_{x\in I}\lvert EMB(x,A)+EMB(M(x),B)\rvert}$
Tetrachords:		4-15 and 4-29, {0146} and {0137}			<001010110000>	<010010010010>	<0−110001000−10>	1
Pentachords:	5-12 and 5-36, {01356} and {01247}				<020221200100>	<111021101110>	<-11-120010-10-10>	0.6
		5-18 and 5-38, {01457} and {01258}			<012210011110>	<101210110120>	<-111000-10010-10>	1
		5-17 and 5-37, {01348} and {03458}			<022200001021>	<102200200021>	<-120000-201000>	1
Hexachords:			6-3 and 6-36, {012356} and {012347}		<243321310100>	<343121111120>	<-10020020-10-20>	0.65
			6-4 and 6-37, {012456} and {012348}		<224422220000>	<342221021021>	<-1-2220120-10-2-1>	0.6
		6-11 and 6-40, {012457} and {012358}			<133221312110>	<231221311130>	<-1020000010-20>	1
				6-10 and 6-39, {013457} and {023458}	<134212121120>	<232211320121>	<-102001-20100-1>	0.8
	6-12 and 6-41, {012467} and {012368}				<121252221110>	<221230242110>	<-1000220-2-1000>	0.8
				6-13 and 6-42, {013467} and {012369}	<044040220220>	<222220220420>	<-222-220000-200>	0.7
		6-6 and 6-38, {012567} and {012378}			<202460222000>	<220460022020>	<0-220002000-20>	1
				6-24 and 6-46, {013468} and {012469}	<032211322121>	<112212321140>	<-12000-10010-21>	0.8
	6-43 and 6-17, {012568} and {012478}				<112430141120>	<112251121121>	<0002-2-102000-1>	1
			6-25 and 6-47, {013568} and {012479}		<030321412130>	<112121413130>	<-12-220000-1000>	0.65
	6-23 and 6-45, {023568} and {023469}				<042020440220>	<122022221420>	<-12000-222-1-200>	0.7
		6-19 and 6-44, {013478} and {012569}			<024420011141>	<104420210141>	<-120000-21000>	1
			6-26 and 6-48, {013578} and {012579}		<020422222040>	<102221423021>	<-12-2201-20-102-1>	0.6
	6-28 and 6-49, {013569} and {013479}				<022221220421>	<024020240240>	<00-22010-202-21>	0.7
				6-29 and 6-50, {023679} and {014679}	<022220222420>	<022040420240>	<0002-20-2022-20>	0.7
Septachords:	7-12 and 7-36, {0123479} and {0123568}				<344241443240>	<253441542230>	<-1-2220120-10-2-1>	0.885
		7-18 and 7-38, {0124569} and {0124578}			<234451431251>	<145451332241>	<1-1-100010-1010>	1
		7-17 and 7-37, {0124569} and {0134578}			<226622421161>	<146622222161>	<1-2000020-1000>	1
Octachords:		8-15 and 8-29, {01234689} and {01235679}			<365673573461>	<356673673451>	<01-1000-100010>	1

Works Cited

Brad Ballinger, Nadia Benbernou, Francisco Gomez-Martin, J. O'Rourke, Godfried Toussaint. "The Continuous Hexachordal Theorem." *Internat. Conf. on Mathematics and Computation in Music (MCM 2009)* 11-21

Capuzzo, Guy. "Maximally Alpha-Like Operations." *Music Theory Online* 14, no. 3 (September 2008).

Lewin, David. “A Response to a Response: On PCSet Relatedness.” *Perspectives of New Music* 18, no. 1 (1979): 498-502.

Patterson, Lindo A.: "Ambiguities in the x-ray analysis of crystal structures." *Physical Review 64*(5-6), 195–201 (1944)

Rahn, John. “Relating Sets” *Perspectives of New Music* 18, no. 1 (1979): 483-498.

Toussaint, Godfried T. . "Interlocking rhythms, duration interval content, cyclotomic sets, and the haxachordal theorem." *Fourth International Workshop on Computational Music Theory*, Universidad Politecnica de Madrid, Escuela Universitaria de Informatica, July 24-28, 2006